Chapter 5 and 6

Section 3: Span and Spanning Sets

(book sections 5.1 and 6.2)

Ideas in this section…

1) The Span of a set of vectors
2) Examples of Span
3) Span is a subspace
4) Is a vector in span?
5) Spanning sets

Span

Def:

1) If $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$ are vectors in vector space V, then $\vec{v} \in V$ is called a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ if there are scalars $a_1, a_2, \dots, a_n \in \mathbb{R}$ such that

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

2) If $\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$ are vectors in vector space V, then the set of all linear combinations of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is called their span and is denoted by

$$span\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \} = \{ a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \mid a_1, a_2, \dots, a_n \in \mathbb{R} \}$$

Notes:

- Each individual vector $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ is in $span(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$
- Any vector you can create by taking some (or all) of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ and combining them by vector addition or scalar multiplication is in $span(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$

Examples of Span

<u>Ex 1</u>: "Find" $span\left\{\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$

Examples of Span

Ex 2: Visualize $span\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$ geometrically

Examples of Span

Ex 3: Visualize $span\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$ geometrically

Examples of Span

Ex 4: Visualize $span\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)$ geometrically

Examples of Span

Ex 5: Visualize $span\left(\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}\right)$ geometrically

Examples of Span

<u>Ex 6</u>: Find some vectors in $span\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \end{bmatrix}\right)$

Then find all vectors in the span, and visualize geometrically

Note: If a vector is added to the list, but is in the span of the previous vectors, then you don't get any new vectors in the new span

Examples of Span

Ex 7: As a subset of $F(-\infty,\infty)$ what is $span(e^x, e^{-x})$

Span is a subspace

Thm 6.2.2: Let V be a vector space, let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$ and let $U = span\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Then…

1) U is a subspace of V

2) U contains each individual vector $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

3) U is the smallest subspace of V containing each of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ (this means that any subspace of V that contains each of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ must contain all of U

Is a given vector in the Span?

Ex 8: Let $\vec{x} = (2, -1, 2, 1)$ and $\vec{y} = (3, 4, -1, 1)$.

a) Is $\vec{p} = (0, -11, 8, 1) \in \text{span}(\vec{x}, \vec{y})$?

Note About Systems of Linear Equations and Matrix Equations

$$\begin{aligned} ax_1 + bx_2 &= e \\ cx_1 + dx_2 &= f \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$x_1 \begin{bmatrix} a \\ c \end{bmatrix} + x_2 \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Are all equivalent!

Is a given vector in the Span?

Ex 8: Let $\vec{x} = (2, -1, 2, 1)$ and $\vec{y} = (3, 4, -1, 1)$.

b) Is $\vec{q} = (2, 3, 1, 2) \in \text{span}(\vec{x}, \vec{y})$?

Is a given vector in the Span?

Ex 9: Let $p_1(x) = 1 + x + 4x^2$ and $p_2(x) = 1 + 5x + x^2$ in P_2. Determine whether $p_1(x)$ and $p_2(x)$ lie in $span\{1 + 2x - x^2, 3 + 5x + 2x^2\}$

Is a given vector in the Span?

Ex 10: Is $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \in span\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ -3 & 2 \end{bmatrix}\right)$?

Is a given vector in the Span?

Ex 11: a) Is $\cos 2x \in span\left(\sin^2 x,\ \cos^2 x\right)$?

Is a given vector in the Span?

Ex 11: b) Is $x^2 \in span(\sin x\ , e^x)$?

Spanning sets

Def: Let V be a vector space. A finite set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a spanning set of V if $span\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = V$.

"Goal": To find a spanning set of a given vector space V (or U !)

Spanning sets

Ex 12a: Show that $\mathbb{R}^3 = span\{ \vec{e}_1 , \vec{e}_2 , \vec{e}_3 \}$

Recall: Note About How To Prove That 2 Sets Are Equal

Def: If A and B are 2 sets, A is a subset of B (denoted $A \subseteq B$) if every element of A is also an element of B.

I.e. $\forall\, x \in A \,,\, x \in B$.

The most common way to show $A = B$ is to show…

1) $A \subseteq B$ and 2) $B \subseteq A$

Spanning sets

Ex 12a: Show that $\mathbb{R}^3 = span\{\, \vec{e}_1 \,, \vec{e}_2 \,, \vec{e}_3 \,\}$

Spanning sets

Ex 12b: Show that $\mathbb{R}^3 = span\{\,(2,-4,2),\ (1,0,1),\ (1,2,3)\,\}$

Spanning sets

Ex 12c: Show that $\mathbb{R}^3 \neq span\{\ (1,2,1),\ (0,1,-1),\ (3,4,5)\ \}$

Spanning sets

Ex 13a: Show that $P_3 = span\{ 1, x, x^2, x^3 \}$

Spanning sets

<u>Ex 13b</u>: Show that $P_3 = span\{\, x^2 + x^3,\ x,\ 2x^2 + 1,\ 3\,\}$

Spanning sets

Ex 14: Find a spanning set for

$$U = \left\{ \begin{bmatrix} 2r & r+3s \\ r-t & 2r+4s+t \end{bmatrix} \mid r, s, t \in \mathbb{R} \right\}$$

Spanning sets

Ex 15: Show that there is no finite set of vectors that span P.

What you need to know from the book

Book reading

Section 5.1 pages 266 - 268
Section 6.2 pages 341 - 343

Problems you need to know how to do from the book

Section 5.1 page 269 #'s 3 - 5, 7 - 12, 16df, 18, 21
Section 6.2 page 343 #'s 6 - 16, 18 - 20, 24, 25, 27